1999 г. № 4
Труды ФОРА

ON THE STOCHASTIC STABILITY OF A NONLINEAR SYSTEM PERTURBED BY A «WHITE» NOISE RANDOM PROCESS

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Summary. In this paper we give sufficient conditions under which one can conclude that the trivial solution of two-dimensional nonlinear system perturbed by «white» noise random process is stochastically stable. The Rayleigh's equation perturbed by «white» noise is given as an example.

1. INTRODUCTION

The present paper in his idea adjoins to work [1]. In [1] Liapunov functions in the of quadratic forms for two- dimensional linear stochastic systems

$$\begin{cases} \dot{x}(t) = (a + e\dot{\xi}(t))x(t) + (b + f\dot{\xi}(t))y(t), \\ \dot{y}(t) = (c + g\dot{\xi}(t))x(t) + (d + h\dot{\xi}(t))y(t), \end{cases} \tag{1.1}$$

where a, b, c, d, e, f, g, h are constants, $\dot{\xi}(t)$ is Gaussian «white» noise random process, were constructed. On the basis on the constructed Liapunov functions in [1] sufficient conditions of stochastic stability of system (1.1) were given.

The main investigation object of the present paper is the two- dimensional system

$$\begin{cases} \dot{x}(t) = f(x(t)) + by(t) + \sigma x(t)\dot{\xi}(t), \\ \dot{y}(t) = cx(t) + dy(t), \end{cases} \tag{1.2}$$

where b, c, d, σ are constants; $f(x)$ is differentiable function $(f\in C^1)$, $f(0)=0$; $x(t)$, $y(t)$ are scalar random processes and is random process of Gaussian «white» noise type.

Our aim is to establish sufficient conditions *of the probability stability, asymptotically stability in the large and quadratic average exponential stability of the trivial solution of system (1.2).*

The deterministic case $\sigma=0$ was considered by Erugin [2] and Malkin [3]. The papers [4] and [5] were dedicated to problems of the construction of Liapunov functions. In [4] Kushner constructed Liapunov functions for linear and certain nonlinear (power nonlinearities type) stochastic differential equations of second order. In [5] the results of [4] were generalized and given sufficient conditions of stochastic stability for certain second order nonlinear (generic equations of nonlinear mechanics) stochastic differential equations.

Note that the definitions of all conceptions and general facts used in this paper from the theory of stochastic differential equations can be found in [6] or [7]. In what follows the system (1.2) is understood *as system of stochastic differential equations in the form of Ito*. By L we shall denote generating differential operator of process $(x(t), y(t))$ for system (1.2), defined as

$$L = (f(x) + by)\frac{\partial}{\partial x} + (cx + dy)\frac{\partial}{\partial y} + \frac{\sigma^2}{2}x^2\frac{\partial^2}{\partial x^2}.$$

2. STATEMENT OF RESULTS

For the system (1.2) the following propositions are true.

Theorem 1. For the given system (1.2) suppose that $f(x)\in C^1(U)$, $U=\{x: |x| <\varepsilon\}$, $\varepsilon>0$, $f(0)=0$ and further:

$$1)\frac{f(x)}{x}+d<0 \quad \forall x\in U, x\neq 0,$$

$$2)d\frac{f(x)}{x}-bc>0 \quad \forall x\in U, x\neq 0,$$

$$3)\left(\frac{f(x)}{x}+d\right)\left(d\frac{f(x)}{x}-bc\right)\leq -\frac{\sigma^2}{2}(d^2+M) \quad \forall x\in U, x\neq 0$$

Where M is such constant that $df'(x)\text{-}bc\leq M \quad \forall x\in U$.

Then the trivial solution of system (1.2) is stable in probability.

Remark 1. It is obvious that the statement of Theorem 1 is remained true, if the conditions 1)-3) replace into ones:

$$1')\frac{f(x)}{x}+d<-\delta<0 \quad \forall x\in U, x\neq 0,$$

$$2')d\frac{f(x)}{x}-bc>\delta_1>0 \quad \forall x\in U, x\neq 0,$$

$$3')\frac{\sigma^2}{2}(d^2+M)\leq \delta_0\delta_1, \quad M=\sup_{x\in U}(df'(x)-bc),$$

respectively.

Theorem 2. Suppose that $f(x)\in C^1(\mathbf{R})$, $f(0)=0$ and the following conditions are satisfied for system (1.2):

1) there is a constant α, such that

$$\alpha+d<0,\ \alpha d-bc>0\ , \tag{2.1}$$

2) there exist constants $\delta_0>0$ and $\delta_1>0$, such that

$$\frac{f(x)}{x}+d<-\delta_0 \quad \forall x\neq 0, \tag{2.2}$$

$$d\frac{f(x)}{x}-bc>\delta_1 \quad \forall x\neq 0, \tag{2.3}$$

$$\alpha+d+\delta_0\geq\frac{\sigma^2}{2}, \tag{2.4}$$

3) there exist a constants M, such that

$$df'(x)\text{-}bc\leq M \qquad \forall x\in R, \tag{2.5}$$

$$\sigma^2/2(d^2+c^2+M)<\delta_0\delta_1. \tag{2.6}$$

Then the trivial solution of system (1.2) is asymptotically stable in the large (with probability 1).

Theorem 3. Suppose that all conditions of Theorems 2 above are true. Assume further that there is a constant $\delta_2>0$ such that

$$d\frac{f(x)}{x}-bc<\delta_2 \quad \forall x\neq 0. \tag{2.7}$$

Then the trivial solution of system (1.2) is exponential stable in quadratic average.

Remark 2. In deterministic linear case, when $\sigma=0$, $f(x)=ax$ the conditions of Theorems 1,2 and 3 turn into necessary and sufficient Routh-Hurwitz ones: $a+d<0$, $ad\text{-}bc>0$.

Remark 3. The investigation of the system

$$\begin{cases} \dot{x}(t)=ax(t)+by(t), \\ \dot{y}(t)=cx(t)+\varphi(y(t))+\sigma y(t)\xi(t) \end{cases} \tag{2.8}$$

reduces to the system (1.2) by change of variables x and into y and x respectively.

Example. We shall consider the Rayleigh's equation $\ddot{x} + F(\dot{x}) + x = 0,$
perturbed by the «white» noise random process $\sigma\dot{x}\dot{\xi}(t)$:

$$\ddot{x} + F(\dot{x}) + x = \sigma\dot{x}\dot{\xi}(t). \tag{2.9}$$

The equation (2.9) reduces to the system of the form (2.8)

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -F(y) - x + \sigma y\dot{\xi}(t). \end{cases} \tag{2.10}$$

By applying the Theorems 1,2 and 3 to (2.10), transformed to form (1.2), we shall get sufficient conditions of probability stability, asymptotically stability in the large and in quadratic average exponential stability of the trivial solution of the system (2.10), hence equation (2.9).

For instance, by using Theorem 2 above we get the following *conditions of asymptotically stability in the large of the trivial solution of the equation* (2.9):

1) $F(y)/y>\delta_0>0 \quad \forall y\neq 0$, 2) $\sigma^2<\delta_0$.

For that it is sufficient to reduce (2.10) into (1.2) by change $x\leftrightarrow y$, then in (1.2) set $f(x)= - F(x)$, $b=-1$, $c=1$, $d=0$.

It should be noted that the condition 2) $\sigma^2<2\delta_0$ above is weaker than corresponding one $\sigma^2<2\delta_0$ in Theorem 2 of paper [5]. The cause is that in the proof of the Theorem 2 [5] a special stochastic Liapunov function was used which in linear case $F(\dot{x}) = b\dot{x}$ it gives necessary and sufficient conditions of asymptotically stability in the large of the trivial solution of the equation (2.9).

3. PROOF OF THE THEOREM 1.

On the basis of the Liapunov function (see [1]) for linear determenistic system ($\sigma=0$, $f(x)=ax$), corresponding to system (1.2), we can construct *Liapunov function for nonlinear system (1.2) as*

$$\frac{1}{2}V(x,y)=(dx-by)^2-bcx^2+2d\int_0^x f(\eta)d\eta. \tag{3.1}$$

We have, by elementary calculation from (3.1), that

$$\frac{1}{2}LV = (f(x)+by)\frac{\partial V}{\partial x} + (cx+dy)\frac{\partial V}{\partial y} + \frac{\sigma^2}{2}x^2\frac{\partial^2 V}{\partial x^2} = 2d^2xf(x) + 2df^2(x) - 2bcxf(x) -$$
$$- 2bcdx^2 + \sigma^2(d^2 + df'(x) - bc)x^2.$$

The expression of $\frac{1}{2}LV$ can be reset in the form

$$\frac{1}{2}LV = 2\left(\frac{f(x)}{x} + d\right)\left(d\frac{f(x)}{x} - bc\right)x^2 + \sigma^2(d^2 + df'(x) - bc), (x\neq 0). \tag{3.2}$$

From (3.2), by using the conditions 1)-3) of Theorem 1, we conclude that $LV\leq 0$ in some pictured neighborhood $\dot{G} = \{(x,y) : |x|\langle\varepsilon, |y|\rangle\varepsilon_1; \mathrm{x}\neq 0, \mathrm{y}\neq 0\}$ of origin.

Positive definiteness of function V follows from the condition 2) of Theorem 1 and the representation of (3.1) as

$$\frac{1}{2}V(x,y) = (dx-by)^2 + 2\int_0^x (df(\eta) - bc\eta)d\eta.$$

By virtue of general stability theorem from [6] the trivial solution of (1.2) is stable in probability. The proof of the Theorem 1 is complete.

4. PROOF OF THEOREM 2.

The main tool for the proof of the Theorem 2 will be *the Liapunov function V(x,y),*defined by

$$V=V_1+V_2,\ ,$$

where

$$\frac{1}{2}V_1=(d^2-bc)x^2+b(b-c)y^2+\alpha dy^2-2bdxy+$$

$$+(cx-\alpha y)^2+2d\int_0^x f(\eta)d\eta,$$

$$V_2=\omega x^2+2\gamma xy+\beta y^2,$$

where α is a constant, placed in the conditions (2.1) and (2.4) of the Theorem 2 ; $\omega>0$, $\beta>0$, $\omega\beta>\gamma$ (that is $V_2>0$). We shall choose the parameters $\omega,\ \beta,\ \gamma$ later. A straightforward calculation yields

$$\frac{1}{2}LV_1=(f(x)+by)\cdot\left[2(d^2-bc)x+2df(x)-2bdy+2c(cx-\alpha y)\right]+$$

$$+(cx+dy)\cdot\left[2b(b-c)y+2\alpha dy-2bdx-2\alpha(cx-\alpha y)\right]+$$

$$+\sigma^2x^2\left[(d^2-bc)+df'(x)+c^2\right]=$$

$$=2\cdot\left\{\left(\frac{f(x)}{x}+d\right)\left(d\frac{f(x)}{x}-bc\right)+c^2\left(\frac{f(x)}{x}-\alpha\right)+\frac{\sigma^2}{2}\left[c^2+d^2+(df'(x)-bc)\right]\right\}\cdot x^2$$

$$+2(\alpha+d)(\alpha d-bc)\cdot y^2+2\alpha c\left(\frac{f(x)}{x}-\alpha\right)xy.$$

Further

$$\frac{1}{2}LV_2=\omega xf(x)+(\gamma c+\omega\sigma^2)x^2+(\omega b+\beta c+\gamma d)xy+(\beta d+\gamma b)y^2+\gamma yf(x).$$

Thus,

$$\frac{1}{2}LV=\left\{\left(\frac{f(x)}{x}+d\right)\left(d\frac{f(x)}{x}-bc\right)+\frac{\sigma^2}{2}\left[c^2+d^2+(df'(x)-bc)\right]\right\}\cdot x^2+$$

$$+(\alpha+d)(\alpha d-bc)y^2+(\beta d+\gamma b)y^2+ \tag{4.1}$$

$$+(\omega b+\beta c+\gamma d-\alpha^2c)xy+(\gamma+\alpha c)yf(x)+\Delta(x),\ x\neq 0,$$

where

$$\Delta(x)=\left[c^2\left(\frac{f(x)}{x}-\alpha\right)+\omega\left(\frac{f(x)}{x}+\frac{\gamma c}{\omega}-\frac{\sigma^2}{2}\right)\right]\cdot x^2,\quad (x\neq 0).$$

We shall choose parameters $\omega,\ \beta,\ \gamma$ such that :

$\Delta(x)\le 0$ for all $x\neq 0$ and

$$\gamma+\alpha c=0,\ \ \omega b+\beta c+\gamma d-\alpha^2c=0,\quad \beta d+\gamma b=0. \tag{4.2}$$

There are two possibilities : 1) $\alpha>0$ and 2) $\alpha<0$.

1) The case $\alpha>0$. From (4.2) we have

$$\gamma=-\alpha c, \tag{4.3}$$

$$\frac{b}{c}\omega+\beta=\alpha(\alpha+d). \tag{4.4}$$

If a) $bc>0$, so since $\alpha(\alpha+d)<0$ there is no positive solution of the equation (4.4).

Let b) $bc<0$. Then the system

$$\begin{cases} \omega\beta > \alpha^2 c^2 \\ \dfrac{b}{c}\omega + \beta = \alpha(\alpha + d) \end{cases}$$

have positive solution (ω,β). From (4.2) we have

$$\beta = \alpha bc / d. \tag{4.5}$$

Further, we find

$$\omega = \frac{\alpha cd(\alpha + d) - \alpha bc^2}{bd} = \frac{\alpha}{d} \cdot \frac{c}{b} \cdot \left[d^2 + (\alpha d - bc)\right]. \tag{4.6}$$

Easily to verify, that $\omega\beta > \alpha^2 c^2 = \gamma^2$.

For chosen values ω and γ we have

$$\frac{f(x)}{x} - \alpha = (\frac{f(x)}{x} + d) - (\alpha + d), \tag{4.7}$$

$$\frac{f(x)}{x} + \frac{\gamma c}{\omega} = (\frac{f(x)}{x} + d) - (d - \frac{\gamma c}{\omega}). \tag{4.8}$$

Easily to check , that

$$d - \frac{\gamma c}{\omega} = d + \frac{\alpha\ c^2}{\omega} < 0.$$

The condition (2.2) of Theorem 2 with regard for (4.7) , (4.8) implies an evaluation

$$\Delta(x) \le -c^2\left[\delta_0 + (\alpha + d)\right]x^2 - \omega\left[\delta_0 + (d + \frac{\alpha c^2}{\omega} - \frac{\sigma^2}{2})\right]x^2. \tag{4.9}$$

Since $0 < \dfrac{d^2}{d^2 + (\alpha d - bc)} < 1$ the inequality (2.4) shall imply $\delta_0 + (d + \dfrac{\alpha c^2}{\omega} - \dfrac{\sigma^2}{2}) > 0.$

Therefore from (4.9) we have $\Delta(x) \le -\left[\dfrac{c^2\sigma^2}{2} - \omega(\alpha + d + \delta_0 - \dfrac{\sigma^2}{2})\right]x^2 \le 0.$

Hence, for the expression (4.1) of $\dfrac{1}{2}LV$ with regard for (4.3) , (4.5) and (4.6) we obtain the estimation ($x \neq 0$):

$$\frac{1}{2}LV \le \left\{\left(\frac{f(x)}{x} + d\right)\left(d\frac{f(x)}{x} - bc\right) + \frac{\sigma^2}{2}\left[c^2 + d^2 + (df'(x) - bc)\right]\right\}x^2 + \\ + (\alpha + d)(\alpha d - bc)y^2 \quad , \tag{4.10}$$

2) The case $\alpha < 0$. By elementary reasoning it is showed, that both in cases a) $bc>0$ and b) $bc<0$ the equation (4.4) have a positive solution $(\omega\ ,\ \beta)$ where ω and β are determined by (4.5) and (4.6) respectively. Therefore as well in considerable case the estimation (4.10) for $\dfrac{1}{2}LV$ holds.

As is obvious from (4.10) the conditions (2.2) , (2.3) , (2.5) and (2.6) of Theorem 2 imply, that LV is negative definite:

$$LV < -k(x^2 + y^2), \tag{4.11}$$

where $k > 0$ is a constant.

Now we show , that the function $V(x, y)$ is positive definite. It sufficient to show positive definiteness of the function V_1 , since $V_2 > 0$. For this we represent the function V_1 in the form

$$\frac{1}{2}V_1 = (dx - by)^2 + (\alpha d - bc)y^2 + (cx - \alpha y)^2 + 2\int_0^x (df(\eta) - bc\eta)d\eta.$$

From here by using the conditions (2.1) and (2.3) we have

$$V_1(x, y) \geq k_1 \cdot (x^2 + y^2), \tag{4.12}$$

where $k_1 > 0$ is a constant.
By taking into account that $V_2(x, y) > 0$ the inequality (4.12) implies the relation

$$\lim_{|x|+|y|\to\infty} V(x, y) = +\infty. \tag{4.13}$$

From correlations (4.11) – (4.13) it follows, that all conditions of general theorem [6] about asymptotically stability in the large are satisfied. The proof of Theorem 2 is complete.

5. PROOF OF THEOREM 3.

It repeats the proof of previous Theorem 2 only by that addition , that the conditions (2.1) , (2.3) and (2.7) imply a double – sided estimation $k_2(x^2 + y^2) \leq V(x, y) \leq k_3(x^2 + y^2)$ for some constants $k_2 > 0$, $k_3 > 0$.

It remains to refer to general theorem from [6]. This complete the proof of Theorem 3.

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МЫЛИНЕЙНЭ СИСТЕМЭ ГОРЭР «МАКЪЭЕФ» ШЭХЭМЫЛЪ ПРОЦЕСС – ПЭРЫКУМ КЪЭЗГЪЭБЫРСЫРЫГЪЭР ЗЭРЭСТОХАСТИК СТАБИЛЫР. (*)

Шумафэ Мыхьамэт Мышъэост ыкъоу

КъеIолIэн. Мы тезисым къеушэты мылинейнэ стохастик системэ горэр дифференциал зэпеIэритIоу зэхэтыр «макъэеф» шIэхэмылъ процесс-пэрыкIум къэзгъэбырсырыгъэр. А системэу зыщIэ къэтIуфгъэм ицыкъоихъохьан пае къэтэгъотых икъухэу стохастикэ кондициехэр сыдыгъуа ар лъэит, асимптотикэ лъэит зэпсэумкIи ыкIи экспоненциал лъэит квадратикэ агуифымкIэ хъура. Щысэ фэдэу Рэлей изэпеIэритIу $\ddot{x} + F(\dot{x}) + x = 0$ «макъэеф» шIэхэмылъ пэрыкIум $\sigma\dot{x}\dot{\xi}(t)$ къэзгъэбырсырыгъэр къэтэты.

О СТОХАСТИЧЕСКОЙ УСТОЙЧИВОСТИ ОДНОЙ НЕЛИНЕЙНОЙ СИСТЕМЫ, ВОЗМУЩЕННОЙ СЛУЧАЙНЫМ ПРОЦЕССОМ ТИПА «БЕЛОГО» ШУМА

М. М. Шумафов

Резюме. В статье рассматривается двумерная нелинейная система дифференциальных уравнений, возмущённая случайным процессом типа «белого» шума. Получены достаточные условия устойчивости по вероятности, асимптотической устойчивости в целом (с вероятностью 1) и экспоненциальной устойчивости в среднем квадратическом нулевого решения рассматриваемой системы. В качестве примера приводится уравнение Рэлея $\ddot{x} + F(\dot{x}) + x = 0$, возмущённое «белым» шумом $\sigma\dot{x}\dot{\xi}(t)$.

(*) I am very grateful to Mr. TSEI ISAK and Miss TSEI NEFEN for their help in the translating this summary into circassian (adyghe language).

