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MODELS AND CONSTRUCTIONS OF GENERATORS $K*(G/H)$ RING ONE CLASSES HOMOGENOUS SPACES G/H

V. Koslov

Armavir State Pedagogical Academy, Armavir

We construct the generators and their Chern characters of the $K^*(SU(N)/SU(2))$ ring, where the $SU(2)$ group is given in irreducible representation. For four cases it is shown their construction. The formulas are given in terms of the Lie groups representation theory.

1 Introduction

Let X be the finite dimensional CW-complex. The set $K(X)$ of equivalence classes of complex vector bundles over the base X is an Abelian group with respect to the Whitney sum. The tensor product bundles determines on $K(X)$ a ring structure. This ring has the unit and called the Grothendieck ring of X complex. With help of Bott's theorem is defined the skew ring $K^*(X)$:

$$K^*(X) = K^0(X) + K^1(X),$$

where $K^0(X)$ is a ring of classes of stably equivalent of complex vector bundles over X base, $K^1(X) = \widetilde{K}^0(SX)$, the $K^1(X)$ is subring of $K^0(X)$, consisting of zero-dimensional elements and SX is the add-on X.

On elements of the $K^*(X)$ ring is determined a place for formula map ch called the Chern character, defines an isomorphism between $K^*(X) \bigotimes Q$ (Q is the rational numbers) and the ring $H^*(X, Q)$ – the rational cohomologyes of space X.

Let now $X = G/H$ be the homogeneous spaces of compact Lie groups G and H. One of the important problems of the complex K-theory is the construction of the elements generators of the ring $K^*(G/H)$. For example, it is necessary in the study of the homotopy invariants. O.V. Manturov in [1] showed that the construction of the generators in $K^*(G/H)$ for a wide class of compact homogeneous spaces is reduced to problems of the representations theory, which can be solved explicitly. But even for the simplest homogeneous spaces present generators is very difficult.

The important problem is the geometrical exposition to within stable equivalence of all complex vectorial bundles over a compact simply connected Lie group. Such problem was considered by O.V. Manturov in [2]. T. Watanabe in [3] for the described Lie groups has constructed the Chern characters of generators elements of $K^*(G)$ ring as exterior algebra with generators $\beta(\rho_\alpha)$, where $\beta : R(G) \to K^*(G)$. The $R(G)$ is the ring of representation of a G group generated by representations ρ_α. For some classes of Lie groups the Dynkins coefficients of corresponding representations and Chern characters are calculated. Outcomes are received with use of works of O.V. Manturov [1], [2].

The complex representation ρ of compact Lie group G leads to complex vectorial stratifications over classifying space BG of G group and further, to the Grothendieck ring $K(BG)$. J.A. Wood in [4] has discovered explicit formulas for Chern characters which have low order terms equal to standard characteristic classes.

In this paper we construction the formulas for generating elements and its Chern characters of $K^*(G/H) \bigotimes Q$ ring in terms of the representations theory. These formulas are obtained in case where $G = SU(N)$, $H = SU(2)$ and the enclosure $SU(2) \to SU(N)$ is given an arbitrary linear irreducible representation of the $SU(2)$ Lie group. In addition, we calculation of the generators and their Chern characters of $K^*(SU(N)/SU(2)) \times Q$ ring for the small dimensions representations of $SU(2)$ group.

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2 Generators $\gamma(\Theta_i)$ and their Chern characters

In our case $X = G/H$, G and H are the compact Lie groups. Then map $f(\Theta) : G/H \to U(n)$, defined by formulae $f(\Theta)gH = \Phi(g)\Psi^{-1}(g)$ one-to-one corresponds (up to homotopy) with element $\gamma(\Theta) \in K^1(G/H)$. Here Φ and Ψ are representations (geometric) of group G, $\dim\Phi = \dim\Psi$, $\Psi^{-1}(g)$ is the matrix inverse to $\Psi(g)$, $\Theta = \Phi - \Psi$ is the virtual representation of group G, which restriction to the subgroup H is equal to zero. Thus the calculating of elements from $K^*(G/H)$ is equivalent to calculating of virtual representations Θ, equaling zero when restricted to the subgroup H.

With the help of representations Θ will be given the system of generators in ring $K^*(SU(N)/SU(2))$ of homogeneous spaces $SU(N)/SU(2)$, where the $SU(2)$ is given with of arbitrary linear irreducible representation φ_k, $\dim\varphi_k = N$, $N = k+1$. Also will be given the Chern characters of these generators.

As it is known, some linear representation φ_k of Lie group is defined by the Dynkin diagram $\overset{k}{\circ}$. The φ_k can be written as a polynom of the variable t (t-representation with diagram $\overset{1}{\circ}$) into ring representations $RSU(2)$ of $SU(2)$ group: $\varphi_k(t) = P_1(t)$.

We considered the system of Adams operations Ψ_1, Ψ_{2^r}, $\Psi_{2^{r+1}}, \ldots$, $\Psi_{2^{r+k-2}}$ as a virtual representations of $SU(N)$ group for sufficiently large r. The restriction $\Psi_1(\varphi_k)$, $\Psi_{2^r}(\varphi_k)$, $\Psi_{2^{r+1}}(\varphi_k), \ldots$, $\Psi_{2^{r+k-2}}(\varphi_k)$ of virtual representations Ψ_1, $\Psi_{2^{r+i}}$, $i = 0, \ldots, k-2$ to the subgroup $SU(2)$ we present in polynomial form in the variable t :

$$\Psi_1(\varphi_k) = P_1(t),\ \Psi_{2^{r+i}}(\varphi_k) = P_{2^{r+i}}(t),\ i = 0, 1, \ldots, k-2$$

and we write the formal difference

$$P_1(t) - \Psi_1,\ P_{2^{r+i}}(t) - \Psi_{2^{r+i}},\ i = 0, 1, \ldots, k-2.$$

The virtual representations

$$\Theta_i = Res_t(P_1(t) - \Psi_1, P_{2^{r+i}}(t) - \Psi_{2^{r+i}}),\ i = 0, 1, \ldots, k-2$$

where Res_t is a result to variable t, are the polynoms from variables Ψ_1, $\Psi_{2^{r+i}}$ are equals zero under restriction to the subgroup $SU(2)$. They also define the elements $\gamma(\Theta_i) \in K^1(SU(N)/SU(2))$. Res_t is the resultant of the variable t.

Теорема 1. *The elements $\gamma(\Theta_0)$, $\gamma(\Theta_1), \ldots \gamma(\Theta_{k-2})$ are the generators system in the $K^*(SU(N)/SU(2)) \bigotimes Q$ ring, where the Lie group $SU(2)$ embedded into $SU(N)$ with arbitrary irreducible representation. The Chern characters of generators $\gamma(\Theta_i)$ have the form*

$$ch\gamma(\Theta_i) = \frac{c_3(f(\Theta_i))x_5}{2!} - \frac{c_4(f(\Theta_i))x_7}{3!} + \ldots + \frac{(-1)^{N-1}c_N(f(\Theta_i))x_{2N-1}}{(N-1)!},$$

where $c_s(f(\Theta_i))$ are the Dynkins coefficients of map $f(\Theta_i)$, which appear under the map cohomology rings $f(\Theta_i) : H^*(SU(N), Q) \to H^*(SU(N)/SU(2), Q)$:

$$c_s(f(\Theta_i)) = (2^{2(r+i)} - 2^{S(r+i)} \cdot (-1)^{k-1}\frac{1}{6}(k^3 + 3k^2 + 2k) \cdot Res_{\widetilde{t}}\left(\frac{\widetilde{P}_1(\widetilde{t})}{\widetilde{t}}, \frac{\widetilde{P}_{2^{r+i}}(\widetilde{t})}{\widetilde{t}}\right),$$

$$S = 3, 4, \ldots, N; i = 1, 2, \ldots, k-2; \textit{for large enough}\, r.$$

Доказательство. Because the group $SU(2)$ is totally nonhomologous to zero in the group $SU(N)$ we apply of Manturov's results [1] (theorem 5.1) $(f^*(\theta)x_1 = 0)$

$$ch\gamma(\Theta_i) = -f^*(\Theta_i)x_3 + \frac{f^*(\Theta_i)x_5}{2!} + \ldots + \frac{(-1)^{N-1}f^*(\Theta_i)x_{2N-1}}{(N-1)!}, \tag{1}$$

where $N = k+1$, $x_3, x_5, \ldots, x_{2N-1}$ — the primitive generators of cohomology ring of $SU(N)$ group. Determination of the $f^*(\Theta_i)x_{2N-1}$ is equivalent calculating of the Dynkins coefficients $c_s(f(\Theta_i))$, $s = 2, 3 \ldots, N$ map $f^*(\Theta_i)$ of the cohomology ring $H^*(SU(N))$ into ring $H^*(SU(N))/SU(2))$.

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The Θ_i is the virtual representation of the $SU(2)$ and it is equal zero in the $RSU(2)$ representations ring. Therefore it is possible to present it in the form of a difference of two parts – positive and negative, having identical dimension and, being already "true"(geometrical) representations.

The Dynkins coefficients of the cohomology Adams operations Ψ_i are calculated under the formula

$$c_s(f(\Theta_i)) = (2^{2(r+i)} - 2^{s(r+i)}) \cdot (-1)^{k-1}\frac{1}{6}(k^3+3k^2+2k)\cdot \cdot Res_{\tilde{t}}\left(\frac{\tilde{P}_1(\tilde{t})}{\tilde{t}}, \frac{\tilde{P}_{2^{r+i}}(\tilde{t})}{\tilde{t}}\right). \tag{2}$$

Then the values of the Chern characters on the elements $\gamma(\Theta_i)$, $i = 0, 1, \ldots, N-3$ are defined by formulas

$$ch\gamma(\Theta_i) = \frac{c_3 f(\Theta_i))x_5}{2!} - \frac{c_4 f(\Theta_i))x_7}{3!} + \ldots + \frac{(-1)^{N-1}c_N(f(\Theta_i))x_{2N-1}}{(N-1)!}, \tag{3}$$

where $x_3, x_5, \ldots, x_{2N-1}$ are the primitive generators of the $H^*(SU(N))/SU(2))$ group and we have the systems of $N-2$ elements of this ring:

$$ch\gamma(\Theta_0), ch\gamma(\Theta_1), ch\gamma(\Theta_2), \ldots, ch\gamma(\Theta_{N-3}). \tag{4}$$

Let's prove, that the system (4) is linearly independent. For this case we will show, that the matrix organized from coefficients $c_s(f(\Theta_i))$, $s = 3, 4, \ldots, N$, $i = 0, 1, \ldots, N-3$ is not degenerated. The determinant

$$\begin{vmatrix} 2^{2r}-2^{3r} & 2^{2r}-2^{4r} & \ldots & 2^{2r}-2^{Nr} \\ 2^{2(r+1)}-2^{3(r+1)} & 2^{2(r+1)}-2^{4(r+1)} & \ldots & 2^{2(r+1)}-2^{N(r+1)} \\ \ldots & \ldots & \ldots & \ldots \\ 2^{2(r+N-3)}-2^{3(r+N-3)} & 2^{2(r+N-3)}-2^{4(r+N-3)} & \ldots & 2^{2(r+N-3)}-2^{N(r+N-3)} \end{vmatrix} \tag{5}$$

is presented in to Vandermont determinants and it have the form to within a sign

$$\begin{gathered} 2^{a_0}(2^r)^{N-3}(2^{r+1})^{N-4}(2^{r+2})^{N-5}\cdot\ldots\cdot 2^{r+N-4}(2-1)(2^2-1)\cdot\ldots\cdot(2^{N-3}-1)+ \\ +2^{a_1}2^{(r+1)^{(N-4)}}\cdot\ldots\cdot 2^{r+N-4}(2^{r+1}-1)(2^{r+2}-1)\cdot\ldots\cdot(2^{r+N-3}-1)+\ldots \\ +2^{a_{N-3}}\cdot 2^{r(N-4)}\cdot 2^{(r+1)(N-5)}\cdot\ldots\cdot(2-1)(2^2-1)\cdot\ldots \\ \cdot(2^{r+N-3}-1)\cdot(1-2^{r+N-5})(1+2^{r+N-4}). \end{gathered} \tag{6}$$

Every item in the sum (6) is even number except one, equal to 1. Such sum is not equal to zero and therefore the system (4) is linearly independent.

The installed linear independence of Chern characters (4) means that the elements

$$\gamma(\Theta_0), \gamma(\Theta_1), \gamma(\Theta_2), \ldots, \gamma(\Theta_{N-3}) \tag{7}$$

are independent linearly .

Really, if some linear combination of the elements (7) is equal to zero, that, taking the same combination of system (4), we obtain the inconsistency with the proved.

The linear independence (4) means also that for every primitive generators x_i of cohomology spaces $SU(N)/SU(2)$ will be such a linear combination of elements (4), that $ch\beta = Dx_i$, where β is the same combination of system (7), D is a rational number. Then, owing to isomorphism

$$ch : K^*(X)\otimes Q \to H^*(X, Q).$$

□

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2.1 The constructions of the generators and the Chern characters

1° **The representation φ_2.** The first nontrivial homogeneous space from the class $SU(N)/SU(2)$ is defined by enclosure φ_2 with Dynkins diagram $\overset{2}{0}$, $\dim\varphi_2 = 3$, $N = 3$, the representation $SU(3)$ have the Dynkins diagram $\overset{1}{0}{-}0$. According to the theorem 1 the system of generators consists one element $\gamma(\Theta_0)$. We will find the Θ_0 and will calculate the $ch\gamma(\Theta_i)$.

In this case it is enough to consider $r = 1$. We take the Adams operations Ψ_1 and Ψ_2, which are considered as virtual representations of the $SU(3)$ group. Then by the definition $\Psi_1 = \Lambda_1$, $\Psi_2 = \Lambda_2^2 - 2\Lambda_2$, where the Λ_1 and Λ_2 are the external degrees of $SU(3)$ group. The restrictions of Ψ_1 and Ψ_2 to $SU(2)$ subgroup are polynoms of the variable $t(t =\overset{1}{0})$:

$$\Psi_1(\varphi_2) = P_1(t), \;\; \Psi_2(\varphi_2) = P_2(t).$$

Let us determine $P_1(t)$ and $P_2(t)$. Obviously $P_1(t) = \Lambda_1(\varphi_2) = \varphi_2(t)$. From Clebsch-Gordan formula we have that $\varphi_1 \otimes \varphi_1 = \varphi_2 + 1$ and then $\varphi_2 = \varphi_2(t)$. Therefore $\Psi_1(\varphi_2) = P_1(t) = t^2 - 1$.

Using the properties of the Adams operations, we find $\Psi_2(\varphi_2)$:

$$\Psi_2(\varphi_2) = P_1, \;\; (\Psi_2(t)) = (t^2 - 2)^2 - 1.$$

Thus we have

$$\begin{aligned} \Psi_1(\varphi_2) = P_1(t) = t^2 - 1, \\ \Psi_2(\varphi_2) = P_2(t) = t^4 - 4t^2 + 3. \end{aligned} \tag{8}$$

Let us make a formal difference

$$\begin{aligned} P_1(t) - \Psi_1 = t^2 - 1 - \Psi_1, \\ P_2(t) - \Psi_2 = t^4 - 4t^2 + 3\Psi_2 \end{aligned} \tag{9}$$

and exclude the variable t from the system (3) with help of the resultant by t :

$$\Theta_0 = Res_t(P_1(t) - \Psi_1, \; P_2(t) - \Psi_2) = [\Psi_1^2 - 2\Psi_1 - \Psi_2]^2 = [\Theta_0']^2.$$

As the restriction to the $SU(2)$ subgroup is equal to zero in the $RSU(2)$ ring, then for the element

$$\Theta_0' = \Psi_1^2 - 2\Psi_1 - \Psi_2 = (\Lambda_1^2 + 2\Lambda_1) - (\Lambda_1^2 + 2\Lambda_1) - 2\Lambda_2 - 2\Lambda_1$$

in the $K^*(SU(3)/SU(2)) \otimes Q$ ring respond the element $\gamma(\Theta_0')$.

By direct calculation we have

$$ch\gamma(\Theta_0') = \frac{c_3(f(\Theta_0'))x_5}{2},$$

where x_5 is the generator element of the chogomology ring of space $(SU(3)/SU(2))$.

Let us find the Dynkins element $c_3(f(\Theta_0'))$. For this we proceed from $\Theta_0 = \Psi_1^2 - 2\Psi_1 - \Psi_2$ to the zero-dimensional representation $\widetilde{\Theta}_0'$ by replacing $\Psi_1 = \widetilde{\Psi_1} + 3$, $\;\Psi_2 = \widetilde{\Psi_2} + 3$, where $\widetilde{\Psi_1} = \Psi_1 - 3$, $\widetilde{\Psi_2} = \Psi_2 - 3$, 3 – the trivial representation of dimension 3.

In that $c_k(f(\Theta_0')) = c_k(f(\widetilde{\Theta}_0')$ and the Dynkin coefficients of tensor multiple of zero-dimensional representations are equal zero, then enough to know only the linear part of the $\widetilde{\Theta}_0'$:

$$\widetilde{\Theta}_0' = \widetilde{\Psi}_1^2 + 4\widetilde{\Psi}_1 - \widetilde{\Psi}_2.$$

Then we have $c_3(f(\Theta_0')) = 4c_3(\Psi_1) - c_3(\Psi_2)$ and $c_3(f(\Theta_0')) = -4$.

Thus we obtain

$$ch\gamma(\Theta_0') = -2x_5. \tag{10}$$

Because the value of $ch\gamma(\Theta_0')$ is nontrivial, that the formula (10) determine isomorphism of rings $K^*(SU(3)/SU(2)) \bigotimes Q$ and $H^*(SU(3)/SU(2)).Q$ $\gamma(\Theta_0')$ is the generators element of $K^*(SU(3)/SU(2)) \bigotimes Q$ ring.

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2° **The representation φ_3.** Let the embedding of $SU(2)$ group is determined now of representation $\varphi_3 : SU(2) \to SU(4)$.

From the Clebsch-Gordan formula we have that $\varphi_2 \otimes \varphi_1 = \varphi_3 + 1$ and then $\varphi_2(t) = t^2 - 1$. Therefore $\varphi_3 = \varphi_3(t) = P_1(t) = t^3 - 2t$.

As $N = dim\varphi_3 = 4$ we have two virtual representations Θ_0 and Θ_1. Take into consideration the values $r = 1$, $k = N - 1 = 3$ find them. The restriction $\Psi_1(\varphi_3)$ of virtual representation Ψ_1 of $SU(4)$ group into $SU(2)$ subgroup have form

$$\Psi_1(\varphi_3) = P_1(t) = t^3 - 2t.$$

By properties of Adams operations we obtain the polynoms

$$\Psi_2(\varphi_3) = P_1(\Psi_2(t)) = P_2(t) = (t^2 - 2)^3 - 2(t^2 - 2),$$

$$\Psi_4(\varphi_3) = P_1(\Psi_2(\Psi_2(t))) = P_4(t) = ((t^2 - 2)^2 - 2)^3 - 2((t^2 - 2)^2 - 2),$$

where $\Psi_2(t) = t^2 - 2$.

Let us make the formal differences

$$P_1(t) - \Psi_1 = t^3 - 2t - \Psi_1,$$
$$P_2(t) - \Psi_2 = t^6 - 6t^4 + 10t^2 - 4 - \Psi_2$$

and calculate the elements Θ_0 and Θ_1 :

$$\Theta_0 = Res_t(P_1(t) - \Psi_1,\ P_4(t) - \Psi_2) = \Psi_1^6 + 7\Psi_1^2\Psi_2 + 8\Psi_1^2 - 3\Psi_1^4\Psi_2 - 8\Psi_1^4 - \Psi_2^3 - 4\Psi_2^2.$$

$$\begin{gathered}\Theta_1 = Res_t(P_1(t) - \Psi_1,\ P_4(t) - \Psi_4) = \Psi_1^{12} + 688\Psi_1^8 + 56\Psi_1^7 + 50843\Psi_1^4\Psi_4 + \Psi_1^8\Psi_4 + \\ +40\Psi_1^3\Psi_4 + 144\Psi_3 + 576 - 32\Psi_1^{10} - 2\Psi_1^9 - 4065\Psi_1^6 - 8\Psi_1^5 - 5732\Psi_1^4 - \\ -160\Psi_1^3 - 3100\Psi_1^2 - 503\Psi_1^2\Psi_4 - 112\Psi_1^6\Psi_4 - 2\Psi_1^5\Psi_4.\end{gathered}$$

The Chern characters of elements $\gamma(\Theta_0)$ and $\gamma(\Theta_1) \in K^1(SU(4)/SU(2))$ have forms

$$ch\gamma(\Theta_0) = \frac{c_3(f(\Theta_0))x_5}{2} - \frac{c_4(f(\Theta_0))x_7}{6},$$

$$ch\gamma(\Theta_1) = \frac{c_3(f(\Theta_1))x_5}{2} - \frac{c_4(f(\Theta_1))x_7}{6},$$

where x_5, x_7 are the primitive generators in ring $H^*(SU(3)/SU(2))$. The Chern characters $ch\gamma(\Theta_0)$ and $ch\gamma(\Theta_1)$ are not proportional. It can be determined directly calculating or from theorem 1.

Now we determine the Dynkins coefficients of maps $f(\Theta_0)$ and $f(\Theta_1)$:

$$c_s(f(\Theta_0)) = (2^2 - 2^s) \cdot 10 \cdot Res_{\widetilde{t}}\Big(\frac{\widetilde{P_1}(\widetilde{t})}{\widetilde{t}},\ \frac{\widetilde{P_2}(\widetilde{t})}{\widetilde{t}}\Big),\ s = 3, 4, \tag{11}$$

$$c_s(f(\Theta_1)) = (2^4 - 2^{2s}) \cdot 10 \cdot Res_{\widetilde{t}}\Big(\frac{\widetilde{P_1}(\widetilde{t})}{\widetilde{t}},\ \frac{\widetilde{P_4}(\widetilde{t})}{\widetilde{t}}\Big),\ s = 3, 4. \tag{12}$$

Let's check up nontrival the eliminants in the formulas (11), (12). The polynomials $P_1(t) - 3$, $P_2(t) - 3$ and $P_1(t) - 3$, $P_4(t) - 3$ pairwise have no general radicals (it is determinate by direct calculations), behind elimination $t = 2$. From here follows, that the pairwise polynomials $\widetilde{P_1}(\widetilde{t})$, $\widetilde{P_2}(\widetilde{t})$ and $\widetilde{P_1}(\widetilde{t})$, $\widetilde{P_4}(\widetilde{t})$ have no common radicals, behind $t = 0$..

Therefore the eliminants in (11) and (12) are not equal to zero. Thus the Chern characters $ch\gamma(\Theta_0)$ and $ch\gamma(\Theta_1)$ are not proportional and are not equal to zero. Then the elements $\gamma(\Theta_0)$ and $\gamma(\Theta_1)$ are the system of generators in the ring $K^*(SU(4)/SU(2)) \bigotimes Q$.

4° **The representation φ_4.** The group $SU(2)$ is realised by representation of φ_4, $dim\varphi_4 = N = 5$; $\varphi_4 : SU(2) \to SU(5)$,. The group $SU(5)$ is set by the elementary representation with Dynkins diagrams $\overset{1}{0}\!-\!0\!-\!0\!-\!0$.

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The Clebsch-Gordan formula of expansion tensor products representations of type A_1 to the direct sum of nonirreducible components have form

$$\varphi_p \otimes \varphi_l = \varphi_{p+l} + \varphi_{p+l-2} + \ldots + \varphi_{p-l},\ P \geqslant l \geqslant 0 \tag{13}$$

In our case from (13) we obtain

$$\varphi_3 \otimes \varphi_1 = \varphi_4 + \varphi_2. \tag{14}$$

With help formula (13) we obtain the representations $\varphi_1(t)$, $\varphi_2(t)$, $\varphi_3(t)$ in polynomials form of variable t :

$$\varphi_1(t) = t,\ \varphi_2(t) = t^2 - 1,\ \varphi_3(t) = t^3 - 2t. \tag{15}$$

From (14) and (15) we obtain that $\varphi_4(t) = t^4 - 3t$.

We will build the system of polynomials $P_1(t)$, $P_2(t)$, $P_4(t)$, $P_8(t)$ using properties of Adams operations Ψ_k :

$$\begin{gathered}\Psi_k(u+\nu) = \Psi_k(u) + \Psi_k(\nu),\\ \Psi_k(\Psi_l(u)) = \Psi_{kl}(u),\\ \Psi_k(u\otimes\nu) = \Psi_k(u)\Psi_k(\nu).\end{gathered} \tag{16}$$

Then

$$\begin{gathered}P_1(t) = \Psi_1(\varphi_4(t)),\\ P_2(t) = \Psi_2(\varphi_4(t)) = P_1(\Psi_2(t)),\\ P_4(t) = \Psi_4(\varphi_4(t)) = P_2(\Psi_2(t)),\\ P_8(t) = \Psi_8(\varphi_4(t)) = P_4(\Psi_2(t)).\end{gathered}$$

As in the $RSU(2)$ ring the representation of the $SU(2)$ groups the generators elements are the exterior degrees $\lambda_0 = 1$ and $\lambda_1 = t$ of the groups $SU(2)$

$$\begin{gathered}\Psi_1(t) = \lambda_1 = t,\\ \Psi_2(t) = \lambda_1^2 - 2 = t^2 - 2,\end{gathered} \tag{17}$$

that

$$\begin{gathered}P_1(t) = t^4 - 3t^3 + 1,\\ P_2(t) = (t^2-2)^4 - 3(t^2-2)^2 + 1,\\ P_4(t) = ((t^2-2)^2-2)^4 - 3((t^2-2)^2-2)^2 + 1,\\ P_8(t) = (((t^2-2)^2-2)^2-2)^4 - 3(((t^2-2)^2-2)^2-2)^2 + 1.\end{gathered} \tag{18}$$

In formulas (18) we make the substitution $u = t^2$. It can be made as k is the even. The result have form

$$\begin{gathered}P_1'(t) = u^2 - 3u + 1,\\ P_2'(t) = (u-2)^4 - 3(u-2)^2 + 1,\\ P_4'(t) = ((u-2)^2-2)^4 - 3((u-2)^2-2)^2 + 1,\\ P_8'(t) = (((u-2)^2-2)^2-2)^4 - 3(((u-2)^2-2)^2-2)^2 + 1.\end{gathered} \tag{19}$$

With the help of eliminant on variable u we build the system of virtual representations Θ_0, Θ_1, Θ_2 of $SU(5)$ group which are converted in zero at restriction on $SU(2)$ subgroup:

$$\begin{gathered}\Theta_0 = Res_u(P_1'(u) - \Psi_1,\ P_2'(u) - \Psi_2),\\ \Theta_1 = Res_u(P_1'(u) - \Psi_1,\ P_4'(u) - \Psi_4),\\ \Theta_2 = Res_u(P_1'(u) - \Psi_1,\ P_8'(u) - \Psi_8).\end{gathered}$$

The Chern characters of elements $\gamma(\Theta_0)$, $\gamma(\Theta_1)$, $\gamma(\Theta_2)$ have form

$$ch\gamma(\Theta_0) = \frac{c_3(f(\Theta_0))x_5}{2} - \frac{c_4(f(\Theta_0))x_7}{6} + \frac{c_5(f(\Theta_0))x_9}{24}, \tag{20}$$

$$ch\gamma(\Theta_1) = \frac{c_3(f(\Theta_1))x_5}{2} - \frac{c_4(f(\Theta_1))x_7}{6} + \frac{c_5(f(\Theta_1))x_9}{24}, \tag{21}$$

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$$ch\gamma(\Theta_2) = \frac{c_3(f(\Theta_2))x_5}{2} - \frac{c_4(f(\Theta_2))x_7}{6} + \frac{c_5(f(\Theta_2))x_9}{24}, \quad (22)$$

where x_5, x_7, x_9 are the primitive generators of the real cohomology ring of homogeneous space $SU(5)/SU(2)$.

Let's show that the elements (20), (21) and (22) are nontrivial and linearly independent elements.

Let's discover the Dynkins coefficients

$$C_s(f(\Theta_0)) = -(2^2 - 2^s) \cdot 5 \cdot Res_{\widetilde{u}}\Big(\frac{\widetilde{P'}_1(\widetilde{u})}{\widetilde{u}}, \frac{\widetilde{P'}_2(\widetilde{u})}{\widetilde{u}}\Big), \ s = 3,4,5, \quad (23)$$

$$C_s(f(\Theta_1)) = -(2^4 - 2^{2s}) \cdot 5 \cdot Res_{\widetilde{u}}\Big(\frac{\widetilde{P'}_1(\widetilde{u})}{\widetilde{u}}, \frac{\widetilde{P'}_4(\widetilde{u})}{\widetilde{u}}\Big), \ s = 3,4,5, \quad (24)$$

$$C_s(f(\Theta_2)) = -(2^6 - 2^{3s}) \cdot 5 \cdot Res_{\widetilde{u}}\Big(\frac{\widetilde{P'}_1(\widetilde{u})}{\widetilde{u}}, \frac{\widetilde{P'}_8(\widetilde{u})}{\widetilde{u}}\Big), \ s = 3,4,5, \quad (25)$$

where $\widetilde{u} = u - 4$.

The polynoms $P'_1 - 5$, $P'_2 - 5$ and also $P'_1 - 5$, $P'_4 - 5$ and $P'_1 - 5$, $P'_8 - 5$ pairwise have no common radicals, except $u = 4$. It is checked by direct calculation. Therefore the eliminants in formulas (23), (24), (25) are not equal to zero.

Thus, the Chern characters (20), (21), (22) are nontrivial. Their linear independence implies from theorem 1 or it can be checked up directly. Also from theorem 1 follows, that the elements $\gamma(\Theta_0)$, $\gamma(\Theta_1)$, $\gamma(\Theta_2)$ of Grothendiecks group are the system of generators in $K^*(SU(5)/SU(2)) \bigotimes Q$ ring.

3 Conclusion

In this paper, we have obtained analytic expressions for the elements $\gamma(\Theta_0)$, $\gamma(\Theta_1), \dots \gamma(\Theta_{k-2})$ and their Chern characters. We have proof that these elements are a system of generators of the $K^*(SU(N)/SU(2)) \bigotimes Q$ ring.

Finally, we note that the calculations of generators and their Chern characters for representations $SU(2)$ group of dimension 3, 4, 5 (and others) make it possible to concluded the restriction on value of r in theorem 1 is excess and $r = 1$.

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Модели и конструкции образующих кольца $K*(G/H)$ класса однородных пространств G/H

В.А.Козлов

Построены модели образующих и их характеры Черна кольца $K^*(SU(N)/SU(2))$, где группа $SU(2)$ дана в неприводимом представлении. Для четырех случаев предъявлены конструкции образующих и характеров Черна. Формулы даны в терминах теории представлений групп Ли.

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