ON THE STABILITY OF A SECOND-ORDER NONLINEAR STOCHASTIC SYSTEM

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The object of the paper is to give sufficient conditions of stochastic stability for certain nonlinear systems of differential equations of order two.

1. Introduction

In this paper we shall proceed the investigation of a system considered in [1]:

$$\begin{cases} \dot{x}(t) = f(x) + by + \sigma x\dot{\xi} \\ \dot{y}(t) = cx + dy, \end{cases} \tag{1.1}$$

where b, c, d and σ are constants, $f(x)$ is differentiable function of x, $f(0) = 0$, $\dot{\xi}$ is random Gaussian "white"noise type process.

We shall be concerned here with the asymptotic stability in the large with probability one (w. p. 1.) and exponential stability in the quadratic average of the trivial solution of the system (1.1). First of all we shall make the following remark in respect of system (1.1).

It was shown in the previous paper [1] that if there are constants $\alpha, \delta_0 > 0, \delta_1 > 0$ such that (I) $\alpha + d < 0, \alpha d - bc > 0$, (II) $f(x)/x + d < -\delta_0$ for all $x \neq 0, df(x)/x - bc > \delta_1$ for all $x \neq 0$ and

$$\alpha + d + \delta_0 \geq \frac{\sigma^2}{2}, \tag{1.2}$$

(III) there exists a constant M such that $df'(x) - bc < M$ for all $x \in R$ and

$$\frac{\sigma^2}{2}(d^2 + c^2 + M) < \delta_0\delta_1, \tag{1.3}$$

then the trivial solution of system (1.1) is asymptotically stable in the large (w. p. 1.). If in addition to conditions (I)-(III) the inequality

$$d\frac{f(x)}{x} - bc < \delta_2 \quad (for \ \ all \ \ x \neq 0)$$

is valid for some constant $\delta_2 > 0$, so the trivial solution of system (1.1) is exponential stable in the quadratic average.

It is easy to see from proof of the theorem 2[1] that if the conditions (1.2) and (1.3) replace into

$$\alpha + d + \delta_0 \geq 0$$

and

$$\frac{\sigma^2}{2}(\beta + d^2 + c^2 + M) < \delta_0\delta_1,$$

where

$$\beta = \frac{\alpha}{d} \cdot \frac{c}{b}[d^2 + (\alpha d - bc)], (d \neq 0),$$

respectively, so the statements of the theorems 2 and 3 of paper [1] remain true.

In the present paper we consider the special case $d < 0$. In this case the conditions of asymptotic stability and exponential stability in the quadratic average given by theorems 2 and 3 from [1] can be loosen.

Note that the deterministic case $\sigma = 0$ was considered by Erugin [2] and Malkin [3].

Later on the system (1.1) is understood as system of stochastic differential equations of Ito's type [4, 5].

2. Statement of the result

The main result to be proved is the following:

Theorem 1. *Suppose that $d < 0$ and that*

1) *there exist a constant α such that*

$$\alpha + d < 0, \quad \alpha d - bc > 0, \tag{2.1}$$

2) *there are constants $\sigma_0 > 0, \sigma_1 > 0$ such that*

$$\frac{f(x)}{x} + d < -\delta_0 \quad for \quad all \quad x \neq 0, \tag{2.2}$$

$$d\frac{f(x)}{x} - bc > \delta_1 \quad for \quad all \quad x \neq 0, \tag{2.3}$$

3) *there is a constant M such that*

$$df'(x) - bc < M \quad for \quad all \quad x \in R \tag{2.4}$$

and

$$\frac{\sigma}{2}(d^2 + M) < \delta_0\delta_1. \tag{2.5}$$

Then the trivial solution of (1.1) *is asymptotically stable in the large (w. p. 1.).*

If in addition to conditions 1)-3) *there is a constant $\delta_2 > 0$ such that the inequality*

$$d\frac{f(x)}{x} - bc < \delta_2 \tag{2.6}$$

is valid, so the trivial solution of system (1.1) *is exponentially stable in the quadratic average.*

Remark. It is clear from comparison of theorem 1 above and theorem 2[1] that the condition (2.5) is weaker than (1.2) and (1.3).

3. Proof of Theorem 1.

Starting with Liapunov function for linear deterministic system ($\sigma := 0, f(x) := ax$) [6], corresponding to (1.1) we construct Liapunov function for stochastic nonlinear system (1.1) in the form

$$V(x,y) = (d^2 - bc)x^2 + (b^2 - \beta bc)y^2 + 2d\int_0^x f(\xi)d\xi + \alpha\beta dy^2 - 2bdxy, \tag{3.1}$$

where α is the constant from (2.1) and β is a positive constant.

By elementary calculation it can be verified that (here L is generating differential operator of process $(x(t), y(t))$ defined by (1.1))

$$\frac{1}{2}LV = x^2P(x) + Q(x,y),$$

where

$$P(x) = \left(\frac{f(x)}{x} + d\right)\left(d\frac{f(x)}{x} - bc\right) + \frac{\sigma^2}{2}[d^2 + (df'(x) - bc)],$$

$$Q(x,y) = \beta(\alpha d - bc)(dy^2 + cxy).$$

The expression for $Q(x,y)$ may be rewritten

$$Q(x,y) = -(\alpha d - bc)[(-\beta d)y^2 - \beta cxy + \gamma x^2] + \gamma(\alpha d - bc)x^2,$$

where the constant $\gamma > 0$ is chosen such that

$$\gamma = \beta\left(\frac{1}{\alpha d - bc} - \frac{c^2}{4d}\right).$$

Then

$$q(x,y) \equiv (-\beta d)y^2 - \beta cxy + \gamma x^2 > 0$$

for all $(x,y) \in R^2 (x \neq 0, y \neq 0)$.

We shall have

$$\frac{1}{2}LV = x^2 P(x) - (\alpha d - bc)q(x,y). \tag{3.2}$$

Choosing $\beta > 0$ sufficiently small (consequently γ) we shall get that the hypotheses (2.2)-(2.5) of the Theorem 1 imply the inequality

$$\frac{\sigma^2}{2}(d^2 + M) + \gamma(\alpha d - bc) \leq \delta_0\delta_1.$$

Hence, by hypothesis (2.1) of the Theorem 1 we have from (3.2)

$$\frac{1}{2}LV \leq -(\alpha d - bc)q(x,y) < 0$$

for all $(x,y) \neq (0,0)$.

Further, we can rewrite the expression (3.1) for V as

$$V = (dx - by)^2 + 2\int_0^x (df(\xi) - bc\xi)d\xi + \beta(\alpha d - bc)y^2.$$

Using the hypotheses (2.1) and (2.3) of Theorem 1 we have that

$$2\int_0^x (df(\xi) - bc\xi)d\xi \geq \delta_1 x^2$$

for all x. Hence, we get

$$V(x,y) \geq \delta_1 x^2 + \beta(\alpha d - bc)y^2. \tag{3.3}$$

From (3.3) by (2.1) we have that

$$V(x,y) \to +\infty \ \ as \ \ x^2 + y^2 \to +\infty.$$

It is also clear that $V(x,y)$ is positive definite in all space R^2.

The last condition (2.6) of Theorem 1 gives that

$$V \leq k(x^2 + y^2)$$

for some constant $k > 0$.

It remains to refer to general theorem from [4]. This completes our proof of Theorem 1.

4. Generalization of Theorem 1.

It can be considered more general system

$$\begin{cases} \dot{x} = f(x) + by + \varphi(x)\dot{\xi} \\ \dot{y} = cx + dy, \end{cases} \tag{4.1}$$

in which b, c, d are constants, and $f(x), \varphi(x)$ are differentiable functions of x, $f(0) = \varphi(0) = 0, \dot{\xi}$ is "white"Gaussian noise.

4.1. *Our general result* is as follows:

Theorem 2. *Assume that* $d < 0$ *and the hypotheses* (2.1)-(2.4) *of the Theorem 1 are true. Suppose that*

$$1)\ \textit{there is a constant } \sigma_0 \textit{ such that } 0 < \frac{\varphi(x)}{x} < \sigma_0 \textit{ for all } x \neq 0$$

$$2)\ \frac{\sigma_0^2}{2}(d^2 + M) < \delta_0\delta_1.$$

Then the trivial solution of system (4.1) *is asymptotically stable in the large (w. p. 1.).*

If in addition to above mentioned conditions the inequality (2.6) *holds, so the trivial solution is exponentially stable in the quadratic average.*

The proof of Theorem 2 is quite similar to the proof of Theorem 1 and is based on using Liapunov function defined by (3.1). Now the expression of LV have the following form

$$LV = 2(f(x) + dx)(df(x) - bcx)x^2 + \varphi^2(x)[d^2 + (df'(x) - bc)] + 2\beta(\alpha d - bc)(dy^2 + cxy).$$

From the conditions of Theorem 2 it follows that LV is negative definite and V is positive definite in all space R^2. Also $V(x, y) \to +\infty$ as $x^2 + y^2 \to +\infty$. Therefore, by [4] the Theorem 2 is established.

Remark. As it is seen from theorems 1 and 2 the problem analogous to Aizerman's one for deterministic system have in considered case positive solution. A namely, it holds the following

Proposition. Let the conditions of Theorem 2 be true. Then if the trivial solution of linear stochastic system

$$\begin{cases} \dot{x} = ax + by + \sigma x\dot{\xi} \\ \dot{y} = cx + dy, \end{cases}$$

is asymptotically stable in the large (w. p. 1.) for all constants $\sigma \in (0, \sigma_0)$, where $\sigma_0 > 0$, so there exist such positive constant $\sigma_* < \sigma_0$ that the trivial solution of nonlinear stochastic system

$$\begin{cases} \dot{x} = ax + by + \varphi(x)\dot{\xi} \\ \dot{y} = cx + dy, \end{cases}$$

where nonlinear function $\varphi(x)$ satisfies the condition

$$0 < \frac{\varphi(x)}{x} < \sigma_* \quad \textit{for all } x \neq 0,$$

is also asymptotically stable in the large (w. p. 1.).

4.2 Now we shall establish *the theorem on the stability in the probability.*

Theorem 3. *Suppose that the following conditions hold in a neighborhood of the origin*

$$1)\ \frac{f(x)}{x} + d \leq 0, \quad (x \neq 0),$$

$$2)\ d\frac{f(x)}{x} - bc > 0, \quad (x \neq 0),$$

$$3)\ \textit{there is a constant } M \textit{ such that } df'(x) - bc \leq M,$$

$$4)\ \textit{there exist a constant } \sigma_0 > 0 \textit{ such that}$$

$$0 < \frac{\varphi(x)}{x} < \sigma_0, \quad (x \neq 0),$$

$$5)\ \left(\frac{f(x)}{x} + d\right)\left(d\frac{f(x)}{x} - bc\right) \leq \frac{\sigma_0^2}{2}(d^2 + M), \quad (x \neq 0).$$

Then the trivial solution of system (4.1) *is stable in the probability.*

Proof. We shall use the Liapunov function defined in [1], a namely we have

$$\frac{1}{2}V(x,y) = (dx-by)^2 + 2d\int_0^x f(\xi)d\xi - bcx^2,$$

$$\frac{1}{2}LV = 2\left(\frac{f(x)}{x}+d\right)\left(d\frac{f(x)}{x}-bc\right)x^2 + \varphi^2(x)[d^2+(df'(x)-bc)] \leq$$
$$\leq 2\left[\left(\frac{f(x)}{x}+d\right)\left(d\frac{f(x)}{x}-bc\right)x^2 + \frac{\sigma_0^2}{2}(d^2+M)\right]x^2.$$

By virtue of the condition 5) of Theorem 3 we obtain the estimate of LV in a neighborhood of the origin

$$LV(x,y) \leq 0.$$

Because $V(x,y)$ is positive definite due condition 2) of Theorem 3 the general theorem [4] completes the proof of Theorem 3.

Remark. In linear case $\varphi(x) \equiv \sigma x$ we obtain the corresponding result from [1].

References

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Мылинейнэу шэпхъитIу зиIэ стохастикэ системэр зэрэстабилыр

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КъеоIолIэн. Мы тезисым къеушэты мылинейнэу стохастик дифференциал зэпеIэритIоу зэхэт системэхэр "макъэеф"шIэхэмылъ пэрыкIум къыгъэбырсырыхэу. А системэхэу зыцIэ къетIуагъэхэм апае къэтэгъотых икъухэу кондициехэр, сыдыгъуа ахэр стохастикэ стабилэу (лъэитэу) хъухэра.

Об устойчивости одной двумерной нелинейной стохастической системы

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В статье для нелинейной стохастической системы второго порядка даны достаточные условия устойчивости по вероятности, асимптотической устойчивости в целом (с вероятностью 1) и экспоненциальной устойчивости в среднем квадратическом.

