REPRESENTATIONS OF LIE ALGEBRAS AND INTEGRATION OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

L. Zh. Palandzhyants

Maikop State Institute of Technology, Maikop

The paper dwells upon the problem of integration of systems of linear differential equations matrixes of with are representations of A_1 Lie algebra. The notion of multiplicative integral is used in this connection.

Let $Y(t)$ be the smooth curve in G Lie group. Then

$$Y' = \lim_{\Delta t \to 0} (Y(t+\Delta t) - Y(t))/\Delta t$$

being an element of a tangent space of G Lie group in point $Y(t) \in G$. Therefore, the element $Y'(t)Y^{-1}(t)$ is of element of a tangent space in neutral element E of G group, that is $Y'(t)Y^{-1}(t)$ belonging to g Lie algebra of G Lie group. We introduce notation $Y'(t)Y^{-1}(t) = A(t)$, where $A(t)$ is a curve in g Lie algebra. Whereby the smooth curve $Y(t)$ in G group is given by the multiplicative integral

$$Y(t) = \overset{\frown}{\int\limits_{t_0}^{t}} E + A(\tau)d\tau. \tag{1}$$

Now let φ be the representation of G group in the $GL(n)$ group of all nonsingular matrices of n order, Φ is the representation of g algebra in $gl(n)$ algebra of all matrices of n order with a zero spur. Then from the equality (1) it follows that the multiplicative integral can be present in $GL(n)$ group. Accordingly the element of integration of matrix function $A(t)$ will be transformed in the $gl(n)$ algebra. Thus, it follows that

$$\varphi(Y(t)) = \overset{\frown}{\int\limits_{t_0}^{t}} E + \Phi(A(\tau))d\tau. \tag{2}$$

If $\varphi = det$, $\Phi = sp$ we get the well-known Liouvilles's formula.

Thus emerge a problem of finding the multiplicative integral in final form from the matrix function of a great order. The question of integration of multiplicative integral in final form is the question of integration of systems of linear differential equations. As it is known the problem of integration of such a system become difficult with growing order of the matrix function. But it can so happen that the multiplicative integral, when it must be represented in its final form is the result of the representation of multiplicative integral of a smaller dimension. In case of homomorvarphism of the groups the conditions of integration in its final form remain stable which enables to compute the multiplicative integral in space of greater dimension. It is only necessery that the initial multiplicative integral be the resultat from representation of certain multiplicative integral of a smaller dimension. Further we shall use the representations of A_1 Lie algebra with the main weigt coefficient $\overset{k}{0}, k \in N$.

We interprete A_1 Lie algebra as an aggregate of quadric matrix with a zero spur. It means that the initial multiplicative integral of a smaller dimension can be represented as:

$$\overset{\frown}{\int} E + A(t)dt, \tag{3}$$

$$A(t) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

where a, b, c are the smooth functions of variable t. The area representation $\overset{k}{0}$ consists of vectors in the form of $E_{-\alpha}{}^{p}\xi,\ p = 0, 1, ..., k$, where ξ is the main vector of representation. We use the standard notations:

$$e_{\alpha} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ e_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$E_{\alpha} = \Phi(e_{\alpha}), E_{-\alpha} = \Phi(e_{-\alpha}),\ H = \Phi(h),$$

The pmatrix $\Phi(A(t))$ we want to find predetermincs the action of operators $E_{\alpha}, E_{-\alpha}, H$ on the basis vectors $E_{-\alpha}{}^{p}\xi$. Using the familiar relations between operators we find to be valid the following expressions:

$$HE_{-\alpha}{}^{p}\xi = (k - 2p)E_{-\alpha}{}^{p}\xi,$$

$$E_{\alpha}E_{-\alpha}{}^{p}\xi = (p(k - p + 1))E_{-\alpha}{}^{p-1}\xi,$$

$$E_{-\alpha}E_{-\alpha}{}^{p}\xi = 1E_{-\alpha}{}^{p+1}\xi,$$

which predermine the matrix elements $\Phi(A(t))$. Thus

$$\Phi(A(t)) = \begin{pmatrix} ak & bk & 0 & \dots & 0 & 0 \\ c & a(k-2) & b(k-2) & \dots & 0 & 0 \\ 0 & c & a(k-4) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & c & (2-k)a & kb \\ 0 & 0 & \dots & 0 & c & -ka \end{pmatrix}$$

In [1] it is shown that the condition $c(t)\int b(t)dt = 2a(t)$ provides the integration of multiplicative integral (3) in its final form. It can be proven that this condition provides integration of the multiplicative integral in its final form with an element of integration matrix function $\Phi(A(t))$. For example, let $k = 2$. Then integrating by parts we get:

$$\overset{\frown}{\int} E + \begin{pmatrix} 2a & 2b & 0 \\ c & 0 & 2b \\ 0 & c & -2a \end{pmatrix} dt = \overset{\frown}{\int} E + \begin{pmatrix} 0 & 2b & 0 \\ 0 & 0 & 2b \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 2a & 0 & 0 \\ c & 0 & 0 \\ 0 & c & -2a \end{pmatrix} dt =$$

$$= \begin{pmatrix} 1 & 2\int b & 2\int^2 b \\ 0 & 1 & 2\int b \\ 0 & 0 & 1 \end{pmatrix} \overset{\frown}{\int} E + \begin{pmatrix} 1 & -2\int b & 2\int^2 b \\ 0 & 1 & -2b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2a & 0 & 0 \\ c & 0 & 0 \\ 0 & c & -2a \end{pmatrix} \begin{pmatrix} 1 & 2\int b & 2\int^2 b \\ 0 & 1 & 2\int b \\ 0 & 0 & 1 \end{pmatrix} dt =$$

$$= \begin{pmatrix} 1 & 2b & 2\int^2 b \\ 0 & 1 & 2\int b \\ 0 & 0 & 1 \end{pmatrix} \overset{\frown}{\int} E + \begin{pmatrix} -2a & 0 & 0 \\ c & 0 & 0 \\ 0 & c & 2a \end{pmatrix} dt =$$

$$= \begin{pmatrix} 1 & 2\int b & 2\int^2 b \\ 0 & 1 & 2\int b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} exp(-\int 2a) & 0 & 0 \\ \int cexp(-\int 2a) & 1 & 0 \\ (exp\int 2a)\int c\int cexp(-\int 2a) & (exp\int 2a)\int c & exp\int 2a \end{pmatrix}.$$

Hence, the multiplicative integral is compute in its final form.

If the element of integration in form of matrix function of greater dimension is not the result of representation of a matrix function of a smaller dimension, then the corresponds system of equations can be reduced to a the linear differential equation $(k + 1)$ order:

$$\nu^{(k+1)} = a_s \nu^{(k-s)},\ s = 1, 2, ..., k.$$

under broad conditions. This equation can be presented in the form of a system with an accompaning matrix, all the superdiagonal elements of which equal 1 while not all the elements of the last row equal zero. Further this metrix can be subjected to a triangular gauge transformation the resultats matrix $(k + 1)$ order is by itself the result of the representation of a certain matrix function of a smaller dimension [2].

References
1. *Palandzhyants L.Zh.* Multiplicative piecemeal integration and Bäcklund transformation. Proceedings of FORA (PSRA), 1996, N 1, 65-75. (in russian)
2. *Palandzhyants L.Zh.* On triangular calibrative transformations. Proceedings of FORA (PSRA), 1997, N 2, 41-43. (in russian)

Представления алгебр Ли и интегрирование систем линейных дифференциальных уравнений

Л. Ж. Паланджянц

В статье рассматривается задача об интегрировании в конечном виде систем линейных дифференциальных уравнений, матрицы которых связаны с представлениями алгебры Ли A_1; для чего используется понятие мультипликативного интеграла

